# Translating First-Order Predicate Logic to Relation Algebra, Implemented using Z3 

Anthony Brogni<br>Department of Computer Science and Engineering<br>University of Minnesota Twin Cities<br>Minneapolis, Minnesota, United States<br>brogn002@umn.edu<br>Sebastiaan J. C. Joosten*<br>Department of Computer Science and Engineering<br>University of Minnesota Twin Cities<br>Minneapolis, Minnesota, United States<br>sjoosten@umn.edu


#### Abstract

This paper presents the development of a software tool that enables the translation of first-order predicate logic into relation algebra. The tool was developed using the Z 3 theorem prover, by leveraging its capabilities to enhance reliability, generate code, and expedite the development process. The resulting standalone Python program allows users to translate first-order logic expressions into relation algebra, eliminating the need to work with relation algebra explicitly. This paper outlines the theoretical background of first-order logic, relation algebra, and the translation process. It also describes the implementation details, including validation of the tool using Z 3 for testing correctness, and discusses deviations from the original translation procedure. By demonstrating the feasibility of utilizing first-order logic as an alternative language for expressing relation algebra, this tool paves the way for integrating first-order logic into tools that traditionally rely on relation algebra as their input language.


[^0]Keywords: first-order logic, relation algebra, Z3, translation, simplification

## 1. Introduction

This paper is about the software development process of a tool that translates first-order logic into relation algebra (and back). In the process, we used the theorem prover Z3, developed by Microsoft Research, to increase reliability, generate part of the code, and speed up development. The final software product is a stand-alone Python tool that does not depend on Z3. This Python implementation addresses an argument that we occasionally encounter in our development of Ampersand [1]: some people choose not to adopt the use of the tool because it would require them to learn how to express their ideas in Relation Algebra. By providing an automatic translation from first-order logic, they do not have to. We believe that some of the ideas used to write this code can be applied in the development of other formal tools as well.

Relation Algebraic operators are often defined in terms of First-Order Logic, so the translation of Relation Algebra into equivalent First-Order Logic terms is well known. It is also known that one can go in the other direction for First-Order Logic terms that concern only binary relations and use no more than three variables. We follow the procedure by Yoshiki Nakamura [2] in our implementation. This procedure outlines how to translate a formula in First-Order Logic with no more than three variables (an FO3 formula) into one in Relation Algebra (RA). However, the domain of all predicates and quantifiers is the same in this translation. Tools that use Relation Algebra as an input language, like Ampersand, but also RelView [3], use heterogeneous Relation Algebra.

Consequently, we adapt the translation procedure to translate FO3 formulas with quantifiers over different domains into well-typed heterogeneous Relation Algebra. The procedure is implemented as a stand-alone Python program that can be run independently so that it can be of use for any tool that takes Relation Algebra as its input language. The implementation is provided as an artifact to this paper [4]. We took care to make the implementation highly readable and to encourage re-implementation into other tools as well. This paper describes the adaptations we needed to make to the procedure to support heterogeneous formulas, as well as how we ensured the correctness of our implementation.

Figure 1 summarizes our development process. The final product consists of three key components: a function to convert FO3 formulas into RA, one to convert back, and a simplification tool for reducing RA formulas. The translation tools are developed manually and its correctness is validated by Z3. In contrast, for the simplification tool, Z3 assumes a more direct role in its creation. We wrote code to generate all potential simplification rules of increasing size, and rely on Z 3 to check their validity, then store the valid rules in a dictionary. Another Python program uses this rule dictionary to generate Python code for simplifying RA formulas. The resulting simplifier is also checked (redundantly) by Z 3 , but its source code is quite large and it would have been tedious to get it right by hand.

All components of this tool come in two flavors: one for homogeneous relation algebra and firstorder logic with only one sort, and one for heterogeneous relation algebra and first-order logic with multiple sorts.


Figure 1. The Development Process. Arrows signify the flow of information. Boxes are software components.

### 1.1. Key Contributions

Key contributions of this paper are:

- An implementation of a translation from FO3 to RA, both for untyped and typed FO3, translating into homogeneous and heterogeneous RA, and back.
- A method of using Z3 during the development of formal tools and in a testing setup.
- A simplifier for RA, which plays a key role in our translation by producing succinct formulas.
- A method for combining Z3 and code generation to write a correct simplifier.


## 2. Theoretical Background

In the following, we assume a global set of non-empty sorts $\mathcal{D}$, a global set of binary predicate symbols $\mathcal{A}$, and a function $d: \mathcal{A} \rightarrow \mathcal{D} \times \mathcal{D}$ that denotes the type of each predicate symbol, where $d_{1}$ and $d_{2}$ are used to denote the two components of $d: d(a)=\left(d_{1}(a), d_{2}(a)\right)$.

An FO3 formula is a formula over the following language, where $a \in \mathcal{A}$ is from a set of binary predicate symbols, $x, y$ are from a set of three variables, and $D \in \mathcal{D}$ is from a set of sorts:

$$
\varphi, \psi \in \mathrm{FO} 3=a(x, y)|x=y| \mathrm{t}|\mathrm{f}| \varphi \vee \psi|\varphi \wedge \psi| \exists x \in D . \varphi|\forall x \in D . \varphi| \neg \varphi
$$

We say that an FO3 formula is closed if every variable occurring in it is bound by a $\forall$ or $\exists$. We say that the type of an occurrence of a variable is the domain $D$ in the respective variable that binds it. We say that an FO3 formula is well-typed if every occurrence of $a(x, y)$ is such that the types of $x$ and $y$
match $d_{1}(a)$ and $d_{2}(a)$ respectively. If $\mathcal{D}$ has precisely one element, we call it the universal set $\mathcal{U}$ and say that the formula is homogeneous. If we place no such restriction on $\mathcal{D}$, we call it heterogeneous.

An RA formula is a formula over the following language, where $a$ is from $\mathcal{A}$, and $D_{1}, D_{2}$ from $\mathcal{D}$ :

$$
\varphi, \psi \in \mathrm{RA}=a\left|\boldsymbol{T}\left[D_{1}, D_{2}\right]\right| \mathbf{0}\left[D_{1}, D_{2}\right]\left|\mathbf{1}\left[D_{1}\right]\right| \varphi \cup \psi|\varphi \cap \psi| \varphi \circ \psi|\varphi \dagger \psi| \bar{\varphi} \mid \varphi^{-1}
$$

We overload $d$ to denote the type of an RA formula by defining it as follows:

$$
\begin{gathered}
d\left(T\left[D_{1}, D_{2}\right]\right)=\left(D_{1}, D_{2}\right), \quad d\left(\mathbf{1}\left[D_{1}\right]\right)=\left(D_{1}, D_{1}\right), \quad d(\varphi \cup \psi)=d(\varphi), \\
d(\varphi \cap \psi)=d(\varphi), \quad d(\varphi \circ \psi)=\left(d_{1}(\varphi), d_{2}(\psi)\right), \quad d(\varphi \dagger \psi)=\left(d_{1}(\varphi), d_{2}(\psi)\right)
\end{gathered}
$$

We say that a formula in RA is well-typed if all of the following hold:

1. Occurrences of $\varphi \cup \psi$ and $\varphi \cap \psi$ satisfy $d(\varphi)=d(\psi)$
2. Occurrences of $\varphi \circ \psi$ and $\varphi \dagger \psi$ satisfy $d_{2}(\varphi)=d_{1}(\psi)$.

To describe the semantics of FO3 and RA, we use an interpretation function $\mathcal{I}$ that takes a closed formula in FO3, a formula in RA, or a domain and produces a Boolean, a set of pairs, or a set, respectively. An interpretation $\mathcal{I}$ maps each expression $\varphi$ to a subset of the Cartesian product $d_{1}(\varphi) \times$ $d_{2}(\varphi)$ such that:

$$
\begin{aligned}
\mathcal{I}\left(\mathbf{0}\left[D_{1}, D_{2}\right]\right) & =\emptyset \\
\mathcal{I}\left(\boldsymbol{T}\left[D_{1}, D_{2}\right]\right) & =D_{1} \times D_{2} \\
\mathcal{I}\left(\mathbf{1}\left[D_{1}\right]\right) & =\left\{(x, x) \mid x \in D_{1}\right\} \\
\mathcal{I}(\bar{\varphi}) & =\left\{(x, y) \mid(x, y) \in d_{1}(\varphi) \times d_{2}(\varphi) \wedge(x, y) \notin \mathcal{I}(\varphi)\right\} \\
\mathcal{I}\left(\varphi^{-1}\right) & =\{(x, y) \mid(y, x) \in \mathcal{I}(\varphi)\} \\
\mathcal{I}(\varphi \cap \psi) & =\{(x, y) \mid(x, y) \in \mathcal{I}(\varphi) \wedge(x, y) \in \mathcal{I}(\psi)\} \\
\mathcal{I}(\varphi \cup \psi) & =\{(x, y) \mid(x, y) \in \mathcal{I}(\varphi) \vee(x, y) \in \mathcal{I}(\psi)\} \\
\mathcal{I}(\varphi \circ \psi) & =\{(x, y) \mid \exists z .(x, z) \in \mathcal{I}(\varphi) \wedge(z, y) \in \mathcal{I}(\psi)\} \\
\mathcal{I}(\varphi \dagger \psi) & =\left\{(x, y) \mid \forall z \in d_{2}(\varphi) .(x, z) \in \mathcal{I}(\varphi) \vee(z, y) \in \mathcal{I}(\psi)\right\}
\end{aligned}
$$

If $\varphi$ is a well-typed closed formula in FO3 and $\psi$ is a well-typed formula in RA, we say that the two are equivalent if $\varphi \Leftrightarrow\left(\mathcal{I}(\psi)=\left\{(x, y) \mid x \in d_{1}(\psi) \wedge y \in d_{2}(\psi)\right\}\right.$ is valid in predicate logic. We say that a translation of $\varphi$ in FO3 into RA is sound if its translation is equivalent.

### 2.1. The Translation Process

In the translation process outlined by Nakamura, there are four steps [2]. The first three steps add properties to the formula that are preserved by any steps that follow them: terms are first put into negation normal form by applying De Morgan's laws. Second, the $\wedge$ and $\vee$ operators are distributed such that each $\exists$ contains a term in conjunctive normal form, and each $\forall$ contains a term in disjunctive normal form. Nakamura calls this a 'good' FO3 formula. Third, the quantifiers are pushed inwards
as much as possible by taking out terms that do not depend on the variable quantified over. The result is a formula where every existential quantifier over, say, $z$, contains a conjunction of terms such that those conjuncts that depend on $x$ do not depend on $y$. Similarly, every universal quantifier contains a disjunction of terms with that property. This is referred to as a 'nice' FO3 formula by Nakamura.

The final step of the translation procedure is to translate a 'nice' FO3 formula into RA. This is the only place where our implementation deviates slightly from the process outlined by Nakamura. We highlight the most important differences later in section 3.5. The translation function we implemented, T , is given as:

$$
\begin{aligned}
& \mathrm{T}(\mathrm{t}):=\boldsymbol{T} \\
& \mathrm{T}(x=x):=\boldsymbol{T} \\
& \mathrm{T}(\neg(x=x)):=\mathbf{0} \\
& \mathrm{T}(a(x, x)):=(a \cap \mathbf{1}) \circ \boldsymbol{T} \\
& \mathrm{T}(a(x, y)):=a \quad \text { if } x \neq y \\
& \mathrm{~T}(\varphi \wedge \psi):=\mathrm{T}(\varphi) \cap \mathrm{T}(\psi) \\
& \mathrm{T}\left(\exists z . \varphi^{\{x, z\}} \wedge \varphi^{\{z\}} \wedge \varphi^{\{z, y\}}\right):=\mathrm{T}\left(\varphi^{\{x, z\}}\right) \circ\left(\mathrm{T}\left(\varphi^{\{z\}}\right) \cap \mathbf{1}\right) \circ \mathrm{T}\left(\psi^{\{z, y\}}\right) \\
& \mathrm{T}\left(\forall z . \varphi^{\{x, z\}} \vee \varphi^{\{z\}} \vee \varphi^{\{z, y\}}\right):=\mathrm{T}\left(\varphi^{\{x, z\}}\right) \dagger\left(\mathrm{T}\left(\varphi^{\{z\}}\right) \cup \overline{\mathbf{1}}\right) \dagger \mathrm{T}\left(\varphi^{\{z, y\}}\right)
\end{aligned}
$$

In the last two lines, $\varphi^{\{x, z\}}, \varphi^{\{z\}}$, and $\varphi^{\{y, z\}}$ stand for the conjuncts (disjuncts) that depend precisely on $x$ and $z, z$, and $y$ and $z$ inside the existential (universal) quantifier, respectively. The type of the identities used in the translation comes from the binding of the corresponding variable.

## 3. Implementation process

Our implementation is written in Python 3.11 and can be found online [4]. We used three important principles in our development: make all code testable early, write single-assignment code, and generate code that cannot reasonably be fully covered with tests.

In creating our tool, our first focus was on making our code testable. For this, we implemented a translation of homogeneous relation algebra to first-order logic, following the interpretation function. We also implemented a - straightforward - translation of first-order logic (in our data structures) into the structures used to represent them in z3. From this point on, all code we wrote could be tested, as outlined in the next section.

As we moved toward heterogeneous translation, we added run-time checks on the datatype for all constructions. That is, whenever a RA formula is constructed, we test at the top level whether it is well-typed. Section 3.2 says a few words about this.

Another principle we used is to write single-assignment code as much as possible. The transformation of FO3 formulas into 'good' and 'nice' FO3 formulas do not (or should not) change the semantics, so one might be tempted to go with a memory-efficient implementation that replaces objects representing terms by their equivalent 'good' and 'nice' versions. We believe this benefit to be quite minor, as code that manipulates formulas is hardly ever part of a critical loop that needs to be heavily optimized. For our use case, the code would be part of a parser of user-written scripts, where
the benefit would be negligible. A major downside of re-assignments is that it makes code much harder to reason about. Unintentional re-assignments did occur at some point and led to bugs (that were immediately caught by our testing). This was due to lists being mutable in Python, a cause that was difficult to spot.

For the most part, our implementation follows the procedure outlined in section 2.1, much of which we implemented as recursive pattern matching on the expression given. All four of the steps mentioned in the previous section are contained within our translation tool. The resulting expressions are sometimes overly verbose. Rather than fixing this by adding special cases to our translation, we decided to add a simplification step to clean up the RA formulas produced. Our simplifier looks at all sub-expressions of a term in RA, and checks if they can be simplified. If so, the simplifier is called again on the simplified term. A problem with writing code for this is that many terms can be simplified, leading to verbose code and a big potential for errors. Moreover, random testing might miss certain terms, especially if a bug is caused by the interplay of several steps. As a consequence, we decided not to write the simplifier by hand, but to generate the code for it instead. This is illustrated in section 3.3 .

### 3.1. Validation with $\mathbf{Z 3}$

To ensure the correctness of our implementation, we conducted automated testing using the z 3 -solver Python module. Our validation process involves the following steps:

1. Random Expression Generation: We utilize a method we created to generate random closed FO3 expressions. This method was written specifically for validation purposes and can be given parameters to specify the size of expression to generate as well as how many of them to generate.
2. Translation to RA: The generated expressions are translated into RA using our implementation of Nakamura's procedure.
3. Simplification of RA: We run tests with and without the simplifier. While building the translations initially, there was no simplifier. Since the simplifier should not change the semantics of RA expressions, this step is optional for the validation.
4. Translation Back: The RA expressions are translated back into FO3. This is the standard direction of the translation, and while it isn't necessary to implement for our tool, it's useful to have it specifically for validation.
5. Equivalence Check: Finally, we use Z 3 to verify if the original expression and the output of the previous step are equivalent. To do so, we ask Z 3 to find a satisfying assignment to the statement that the formulas are not equivalent. If Z 3 finds such a counterexample, it indicates an error in our implementation.

To create a flexible and robust testing system, we generated random closed FO3 formulas of various specified sizes (number of symbols). This approach enabled us to validate our code thoroughly. In practice, our method of validation helped us to discover and eliminate initial bugs that were present in early versions of our software tool.

Adapting our software tool to include support for translating heterogeneous formulas became significantly easier with the assistance of Z3. Leveraging Z3, we were able to employ a trial-and-error approach for some parts of the project by initially making educated guesses about the code, testing it with Z 3 , and subsequently analyzing the results to determine the correctness or incorrectness of the written code. This facilitated a more effective development process.

We observed that for many equivalent formulas, Z 3 can prove their equivalence very quickly. To use this, we run Z 3 with a low timeout initially (about 60 ms ). If Z 3 times out and returns 'unknown', we have a high chance that the formula is false, and we rerun the query on a finite sort with four elements and a larger timeout (about 1s): Many non-equivalent formulas are quickly disproved with a small finite model. In this case, if Z3 returns 'unsat' (it was not able to find a counterexample) or 'unknown', we rerun the query again, using a very large timeout (minutes). This last case rarely occurs in practice.

### 3.2. Type Checking

Upon creation of heterogeneous RA objects, we check to ensure that the properties of a well-typed formula defined in section 2 hold. If the properties hold, then the heterogeneous RA formula is welltyped. If one (or more) of the properties is broken, however, then our code raises an exception to alert the user of our tool that an ill-typed RA formula has been created. Errors caught using this check were minor typos that were easy to fix. It is hard to say whether they would have been as easy to fix if these checks were not in place.

While an end-user should never have to experience such exceptions while using our tool, the constructors of RA terms are considered 'public', so we left these safety checks in place. This way, users that use these constructors directly will be alerted of ill-typed formulas if they occur.

### 3.3. Simplification of RA Formulas

Building upon the concepts used in our automated testing implementation, we also developed a reliable tool for simplifying RA formulas. The code to simplify RA formulas is generated by a separate Python program.

The first step in building our simplifier was to enumerate all formulas that can be considered simplifications for homogeneous RA formulas. To do so, we wrote code that could generate all formulas of a specific size, similar to the code that randomly creates formulas of a certain size. Here we used the yield keyword in Python to generate all possible formulas of that size. This produces an iterator that generates all terms, and the resulting code looks similar to what one would write using a list monad for this in a functional programming language.

To come up with all simplifying rules, we generate pairs of RA formulas $(\varphi, \psi)$, where $\varphi$ is of a specified size and $\psi$ is smaller than it. Moreover, we require that relation symbols in $\psi$ must occur at least as often in $\varphi$. As a consequence, replacing a term $\varphi$ by $\psi$ in the way of term rewriting would reduce the size of the overall term. From the generated rules, we select only those that are valid by checking their equality with Z3. The valid rules are then stored in a dictionary that is saved to a file for future use.

We generate all simplifying rules starting at size 1 and increase this size until generating all rules takes too much time. To cut down on the number of rules that need to be tested, we require that $\varphi$ and $\psi$ cannot be simplified by any of the rules that were generated in earlier iterations. To check this, we simply call the simplifier on $\varphi$ and $\psi$.

From the dictionary of simplification rules, we generate Python code for simplifying RA expressions. The generation process begins by writing code that matches $\varphi$ and replaces it with $\psi$. An example is given in Figure 2. The code is generated such that blocks with similar prefixes can be grouped, by doing so the resulting code is more efficient (because patterns are only matched once) and concise. The grouping is simply done by finding common string prefixes using a function ('group_by_prefix'), after which it is simply written to a file (by 'write_grouped_code').

```
def simplify(expression):
    if isinstance(expression, COR_Expressions. Intersection ):
    lhs1, rhs1= expression.argument1, expression.argument2
    if isinstance(lhs1, COR_Expressions. Union):
            lhs2, rhs2= lhs1.argument1, lhs1.argument2
            A= lhs2
            B}= rhs
            if str(B)== str(rhs1):
                return ("((A)\cup(B))\cap(B)=B", B )
```

Figure 2. $\quad$ Simplification code generated for the rule $((A) \cup(B)) \cap(B)=B$

The generated simplifier matches the term at the top level and returns the simplified term along with the rule that was used to perform the simplification. This is done so we can use the simplifier to eliminate redundant rules and eliminate dead code in the generated files. For instance, the rules $A \cap A \rightarrow A$ and $B \cap B \rightarrow B$ both match on precisely the same terms. However, when $B \cap B$ is provided to the simplifier, the returned rule is $A \cap A \rightarrow A$, indicating that the rule $B \cap B \rightarrow B$ is redundant.

To find all heterogeneous RA rules, we observe that any well-typed rule that is valid remains valid if all domains are set to $\mathcal{U}$. This means that every heterogeneous RA rule is represented by a homogeneous one. To find all heterogeneous rules, we simply add types to the homogeneous ones and test them for well-typedness. If the resulting rule is well-typed, the equivalence check in Z 3 is performed as well. The resulting heterogeneous rules were then run through the simplifier to remove redundant rules.

### 3.4. Note on our Use of $\mathbf{Z 3}$

The role of Z3 in this project is threefold:

1. We use it to validate our code via the role it plays in testing.
2. We use it during code generation to remove incorrect code and leave the correct code.
3. We use it as an argument that using relation algebra can be a useful alternative to reasoning in first-order logic.

None of these roles require the end users of our code to run Z3. The tool we developed incorporates the results produced using Z 3 , but not Z 3 itself.

Z 3 is used to validate our translation tool, and in turn, we can use our translation tool to evaluate Z3's effectiveness as a theorem prover. While testing our tool, we sometimes came across equations that Z 3 was unable to prove or disprove. They can be proved by Z 3 for a finite sort, but otherwise time out (after 10 minutes). These equations are almost always quite large. The following is an example of one such equation:

$$
\begin{aligned}
& \forall y \cdot \forall z \cdot(\exists x \cdot(\exists y \cdot(B(y, x)) \wedge(B(x, z)))\vee(\neg(\neg(C(x, y))))) \wedge((C(y, y)) \vee(\text { True })) \\
& \stackrel{?}{=} \\
&(\forall y \cdot \exists z \cdot(B(z, y)) \wedge(\exists y \cdot B(y, z))) \vee(\forall z \cdot \exists y \cdot C(y, z))
\end{aligned}
$$

It is significant that Z 3 times out on this question because the question is an FO3 formula. Translating this formula to RA and simplifying it using our simplifier results in a trivially true formula. This whole process takes a couple of milliseconds. This means that Z 3 itself could be improved by using our translation to RA. Moreover, it indicates that combining Z3 and tools like it with a tool that performs equational reasoning in RA has the potential to improve Z 3 significantly.

### 3.5. Deviations from the Original Procedure

In the original paper by Nakamura [2], the translation of $R(x, y)$ as described is only sound if $x$ and $y$ are different variables. Fortunately, there is a note in the paper (Remark 2) that describes exactly what to do, which is to use the logical equivalence $R(x, x) \equiv \exists y . R(x, y) \wedge x=y$. This translates $R(x, x)$ into RA as follows: $(R \cap \mathbf{1}) \circ \boldsymbol{T}$. Types can be added to $\mathbf{1}$ and $\boldsymbol{T}$ according to the type of the occurrence of $x$.

Relatedly, in Nakamura's paper [2], the expression $\exists z . \varphi_{1}(x, z) \wedge \varphi_{2}(z, y)$ translates to $P_{1} \circ P_{2}$, where $P_{1}$ and $P_{2}$ are the translations of $\varphi_{1}$ and $\varphi_{2}$ respectively. Unfortunately, this does not necessarily produce well-typed RA formulas. The issue occurs in a sub-expression like: $\exists z \in A . a(z, z)$ as a set of type $A \times B$. Its translation would be $a \cap \mathbf{1}[A]$, which is a set of type $A \times A$.

To mitigate the issue, we treat the conjuncts that only depend on a single variable separately: $\exists z . \varphi_{1}(x, z) \wedge \varphi_{2}(z) \wedge \varphi_{3}(z, y)$ is translated to $P_{1} \circ\left(P_{2} \cap \mathbf{1}\right) \circ P_{3}$, where again $P_{i}$ is the translation of $\varphi_{i}$ for $i \in\{1,2,3\}$. The translation of $\forall$ quantifiers with a dagger is treated similarly: $\forall z . \varphi_{1}(x, z) \vee$ $\varphi_{2}(z) \vee \varphi_{3}(z, y)$ translates to $P_{1} \dagger\left(P_{2} \cup \overline{\mathbf{1}}\right) \dagger P_{3}$.

The translation, as originally given [2] and as presented in this paper, talks about FO3 as an expression language where only three variables can occur ( $x, y$ are variables taken from a set with precisely three elements when we introduced FO3). In practice, the author of an FO3 expression might like to use meaningful variable names. As such, our implementation translates any expression
that is alpha-equivalent to an FO3 expression. Take the following first-order logic expression as an example:

$$
\exists x \in A . \exists y \in B . \exists z \in C . \exists w \in D . a(x, w)
$$

Its translation would be:

$$
\boldsymbol{T}_{[L e f t \times A]} \circ a \circ \boldsymbol{T}_{[D \times r i g h t]}
$$

### 3.6. Size of the Project

Our project consists of 2338 lines of code typed by hand and 3072 lines of simplification code generated with the help of Z 3 . Of these 5410 combined lines of code, 4254 are part of the final product that an end user would interact with, while the remaining 1156 lines were used only to create the final product.

Another interesting statistic is that our rule dictionaries currently contain 250 homogeneous simplification rules and 263 heterogeneous simplification rules, meaning 13 of our heterogeneous rules cannot be generalized by another typing of the same rule. For example, the following are two different typings of the same homogeneous simplification rule, but neither one generalizes the other:

$$
\begin{gathered}
((A[P, P]) \circ(B[P, Q])) \cup((A[P, P]) \circ(C[P, Q]))=(A[P, P]) \circ((B[P, Q]) \cup(C[P, Q])) \\
\quad \text { and } \\
((A[P, Q]) \circ(B[Q, Q])) \cup((A[P, Q]) \circ(C[Q, Q]))=(A[P, Q]) \circ((B[Q, Q]) \cup(C[Q, Q]))
\end{gathered}
$$

## 4. Conclusion and Future Work

This project succeeded in implementing a translation from FO3 to RA, and back. We addressed FO3 terms with repeated arguments, as well as typed FO3 formulas and heterogeneous Relation Algebra, which is to the best of our knowledge a new contribution. Furthermore, we extensively tested our implementation using Z3 and our type-checking system.

Some details still need to be figured out before people can write First-Order Logic formulas as an alternative to writing valid Ampersand or RelView code. These details include minor syntax issues and the fact that neither Ampersand nor RelView is built in Python. In the case of Ampersand [1], the heterogeneous Relation Algebra used allows for sub-typing: if A is a subset of B , then a relation R from A to A can be composed with a relation $S$ from $B$ to $B$. While this should relax the language in principle, we need to confirm that translations in these cases are as one would expect them to be.

A second question is how to get around the issue that only three variables are allowed in FO3 terms. While three variables suffice for many declarations, one would occasionally hit this limit, which might be quite a surprise to the unwary user. Introducing additional predicates can help get around this limit, as shown by Givant [5]. Doing this practically means that we do not wish to introduce too many predicates. Moreover, we wish their values could be calculated in a reasonably efficient way in systems like Ampersand or RelView. How to achieve this is an open question.

We have discovered through our work that there are some FO3 equations that Z3 is unable to prove nor disprove the validity of. It would be interesting to look further into why this is and how Z 3 could be improved to quickly prove or disprove such equations.

## References

[1] Joosten SMM, Joosten SJC. Type checking by domain analysis in ampersand. In: International Conference on Relational and Algebraic Methods in Computer Science. Springer, 2015 pp. 225-240.
[2] Nakamura Y. Expressive Power and Succinctness of the Positive Calculus of Relations. In: International Conference on Relational and Algebraic Methods in Computer Science. Springer, 2020 pp. 204-220.
[3] Berghammer R, Neumann F. RelView-An OBDD-based Computer Algebra system for relations. In: International Workshop on Computer Algebra in Scientific Computing. Springer, 2005 pp. 40-51.
[4] Brogni A, Joosten SJC. Translating First-Order Predicate Logic to Relation Algebra. Available online at https://doi. org/10. 5281/zenodo. 8185085, 2023.
[5] Givant S. The calculus of relations as a foundation for mathematics. Journal of Automated Reasoning, 2006. 37(4):277-322.


[^0]:    *Address for correspondence: sjoosten@umn.edu

